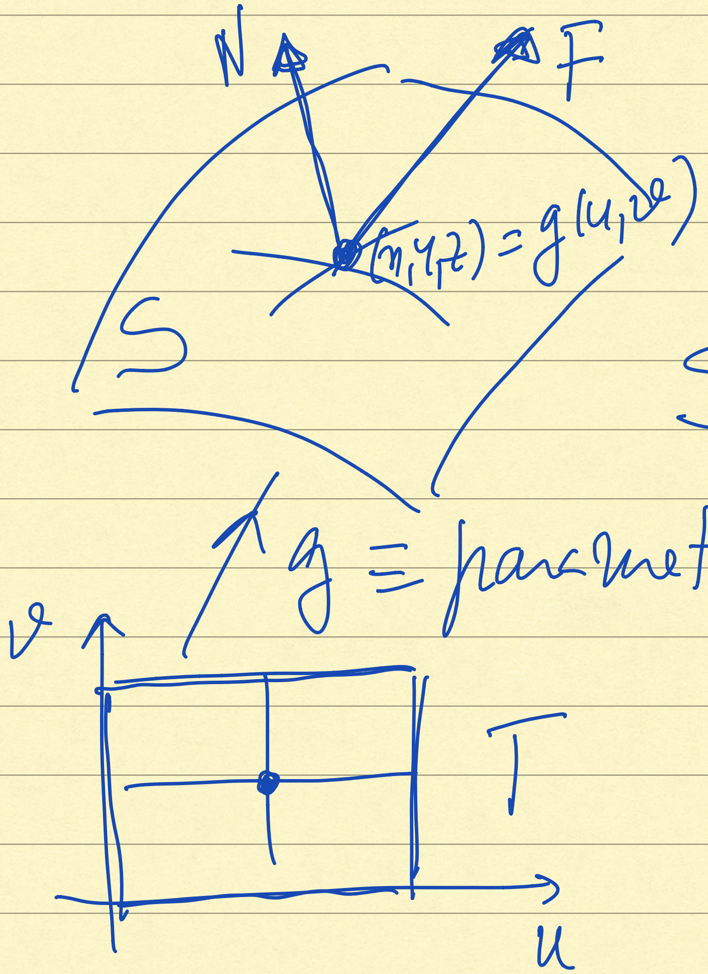
Fluxo. Teorema da Divergência

$F : \mathbb{R}^3 \to \mathbb{R}^3$, $S \subset \mathbb{R}^3$ superfície



$\|N\| = 1$

$S = g(T)$

g = parametrização de S.

$$\int_S F\cdot N = \iint_S F\cdot N$$

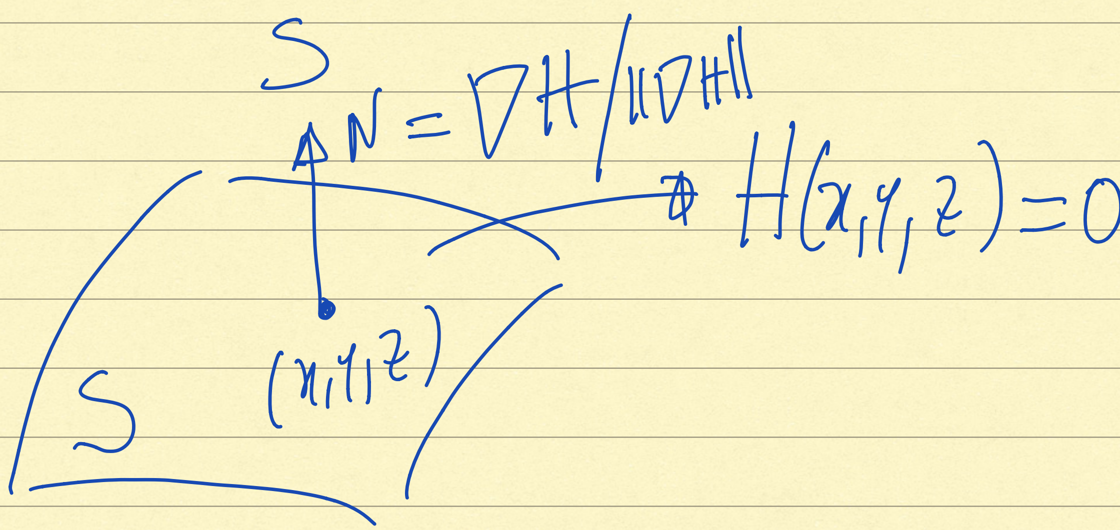
Fluxo de F através de S no sentido da normal N.

$$\iint_S F\cdot N = \pm \iint_T F(g(u,v))\cdot \underbrace{(D_u g \times D_v g)}_{\text{Normal}}\,du\,dv$$

$$\parallel$$

$$\iint_S F\cdot N \quad , \text{ie} \quad F\cdot N = k \ (\text{constante})$$

$$= \iint_S k = k \,\mathrm{Vol}_2(S).$$



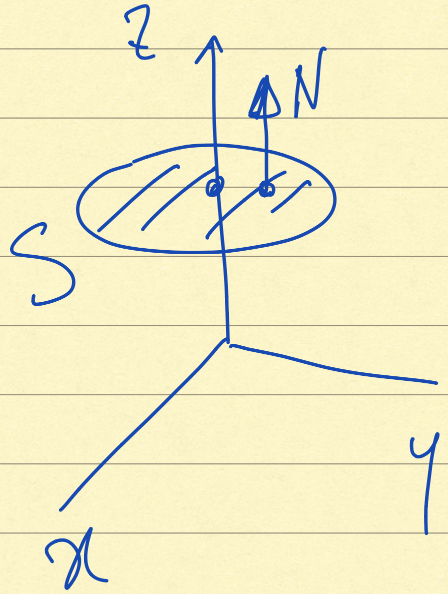
Exemplo: $F(x,y,z) = (x,y,z)$

$S: \boxed{z=1}\ ;\ x^2+y^2<1.$

$N: N_z > 0.$

$H(x,y,z) = z-1 = 0$ em S.

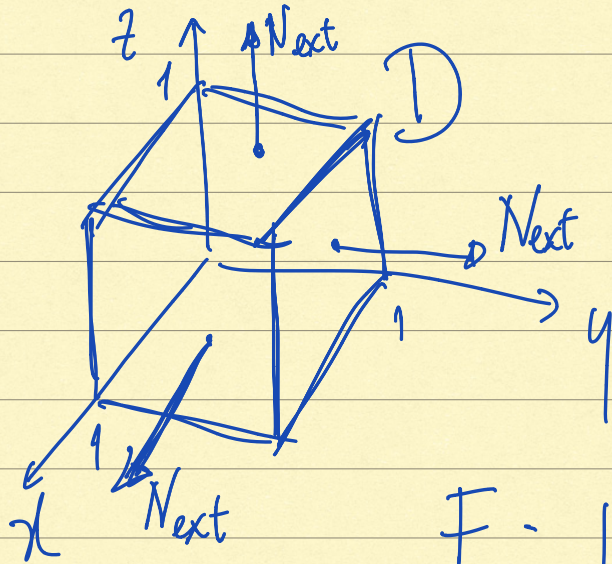
$$N = \frac{\nabla H}{\|\nabla H\|} = \frac{(0,0,1)}{1} = (0,0,1)$$



$$\iint_S F\cdot N = \iint_S z = \iint_S 1$$

$$= Vol_2(S)$$

$$= \pi \checkmark$$



$0 < x < 1$

$0 < y < 1$

$0 < z < 1$

aberto

limitado

$F : \mathbb{R}^3 \to \mathbb{R}^3$

$F = (P, Q, R)$

$$\iint_{\partial D} F \cdot N_{ext} = \text{Soma de 6 fluxos}$$

1) $S_1 = \{ (x, y, z) : \boxed{x = 0} ; \; 0 < y < 1, \; 0 < z < 1 \}$

$N_{ext} = (-1, 0, 0)$

$F(0, y, z) = \Big(P(0, y, z), Q(0, y, z), R(0, y, z) \Big)$

$F \cdot N_{ext} = -P(0, y, z)$

$$\iint_{S_1} F \cdot N_{ext} = -\iint_{S_1} P(0,y,z)$$

$$= -\int_0^1 \int_0^1 P(0,y,z)\, dy\, dz$$

$$S_2 = \{ (x,y,z) : \boxed{x=1};\ 0<y<1,\ 0<z<1 \}$$

$$N_{ext} = (1,0,0)$$

$$F(1,y,z) = \big(P(1,y,z), Q(1,y,z), R(1,y,z)\big)$$

$$F \cdot N_{ext} = P(1,y,z)$$

$$\iint_{S_2} F \cdot N_{ext} = \int_0^1 \int_0^1 P(1,y,z)\, dy\, dz$$

S_3 : $y=0$; $0<x<1$; $0<z<1$

$N_{ext} = (0,-1,0)$

$$\iint_{S_3} F \cdot N_{ext} = -\int_0^1 \int_0^1 Q(x,0,z)\,dx\,dz$$

S_4 : $y=1$; $0<x<1$; $0<z<1$

$N_{ext} = (0,1,0)$

$$\iint_{S_4} F \cdot N_{ext} = \int_0^1 \int_0^1 Q(x,1,z)\,dx\,dz$$

S_5 : $z=0$; $0<x<1$; $0<y<1$

$N_{ext} = (0,0,-1)$

$$\iint_{S_5} F\cdot N_{ext} = -\int_0^1\int_0^1 R(x,y,0)\,dx\,dy$$

S_6 : $z=1$; $0<x<1$; $0<y<1$

$$N_{ext} = (0,0,1)$$

$$\iint_{S_6} F\cdot N_{ext} = \int_0^1\int_0^1 R(x,y,1)\,dx\,dy$$

$$\iint_{\partial D} F\cdot N_{ext} = \int_0^1\int_0^1 \Big(P(1,y,z)-P(0,y,z)\Big)\,dy\,dz$$

$$+\int_0^1\int_0^1 \Big(Q(x,1,z)-Q(x,0,z)\Big)\,\underline{dx\,dz}$$

$$+$$

$$+ \int_0^1 \int_0^1 \left(R(x,y,1) - R(x,y,0) \right) dx\, dy$$

$$= \int_0^1 \int_0^1 \left(\int_0^1 \frac{\partial P}{\partial x}(x,y,z)\, dx \right) dy\, dz$$

$$+ \int_0^1 \int_0^1 \left(\int_0^1 \frac{\partial Q}{\partial y}(x,y,z)\, dy \right) dx\, dz$$

$$+ \int_0^1 \int_0^1 \left(\int_0^1 \frac{\partial R}{\partial z}(x,y,z)\, dz \right) dx\, dy$$

+ Fubini em D

$$\boxed{\iint_{\partial D} F\cdot N_{ext} = \iiint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx\,dy\,dz}$$

$$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$\equiv$ divergência de F.

$$\boxed{\iint_{\partial D} F\cdot N_{ext} = \iiint_D \underbrace{\operatorname{div} F}}$$

Teorema da divergência

$F: \mathbb{R}^3 \to \mathbb{R}^3$, C^1 em D.

$D \subset \mathbb{R}^3$, aberto, limitado,
Regular ($\partial D \equiv$ superfície C^1).

$F: \mathbb{R}^3 \to \mathbb{R}^3$, C^1 em D

$\operatorname{div} F : \mathbb{R}^3 \to \mathbb{R}$.

$$\operatorname{div} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

$$\iiint_D \operatorname{div} F = \underbrace{\iint_{\partial D} F \cdot N_{ext}}_{\boxed{\text{FLUXO}}}$$